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RESEARCH MEMORANDUM

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RESEARCH MEMORANDUM

A STUDY TO DETERMINE THE FLIGHT PATHS WHICH REQUIRE MINIMUM TIME AND MINIMUM FUEL FOR A TYPICAL PRESENT DAY INTERCEPTOR (U)

by

T. F. Kirkwood

RM-246

September 15, 1949

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SUMMARY

An approximate method is presented for finding the flight technique by which an airplane may reach a specified speed and altitude in the minimum time or with the minimum fuel consumed. Both the case in which the distance covered is unrestricted and in which a specified distance is covered are considered.

It is found, that if the final altitude is sufficiently far above the initial, a large portion of the optimum path is independent of the specified initial and final speeds and altitudes. Simple equations are found for this portion of the path. No equations are found describing the transition from the initial speed and altitude to this central portion of the path or from the central portion to the final speed and altitude. However, some information regarding the best transition technique is found from a consideration of several arbitrarily selected transition paths.

1. INTRODUCTION

The purpose of this report is to present methods for rapidly discovering the flight technique by which an interceptor may climb from take-off to a specified speed and altitude in the minimum time or with minimum fuel consumed. Both the case in which a specified distance is covered and the case in which the distance is not specified are considered. In each case techniques for minimum time and minimum fuel are developed. All the cases have been investigated assuming that the throttle setting remains constant throughout the flight. It is found that this procedure is optimum for all cases except that of minimum fuel with a specified distance greater than the distance covered in climb.

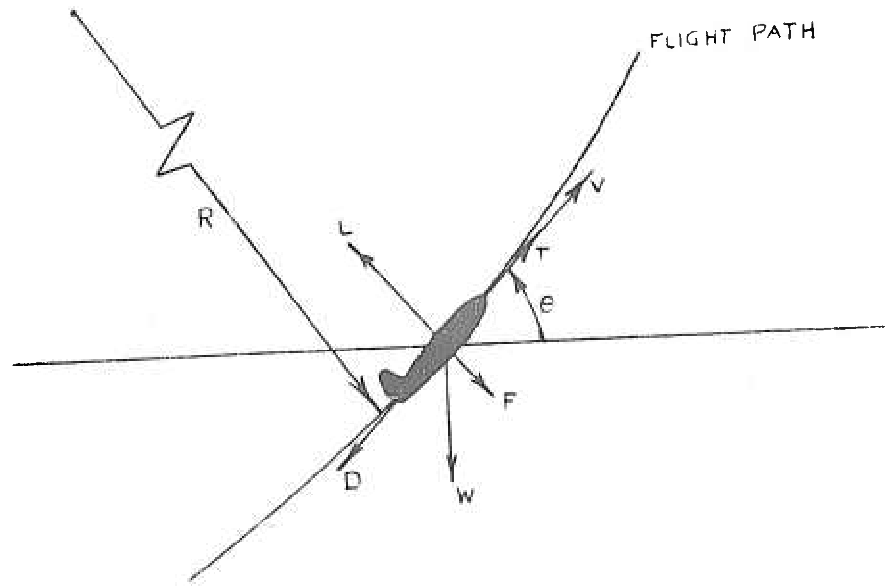
Two methods are used to discover these techniques. The first method is to attempt a direct solution by means of the calculus of variations. It is soon found that the resulting equations are too complicated to be used in ordinary engineering work. Consequently, a trial and error method is used in which arbitrary paths of speed vs. altitude are selected and the time and fuel along them calculated. A study of the results of these calculations indicates that if the final altitude is sufficiently far above the initial altitude, the greater portion of the optimum flight path lies along a speed-altitude path which is independent of the specified initial and final speeds and altitudes. This path, which is characteristic of the airplane design, but does not depend upon the initial and final speeds and altitudes, will be referred to as the "characteristic path". An optimum flight between specified speed - altitude points consists of a transition from the initial point to the characteristic path, a climb up the characteristic path, and a transition from the characteristic path to the final point.

After the existence of the characteristic path has been suggested by the trial and error method, it is found possible to develop simple formulas for it by a mathematical approach. In addition to supplying a direct method of discovering the characteristic path for any airplane, these formulas show that the characteristic path depends upon the specified distance covered in the flight, the throttle setting used during the flight, and also on whether time or fuel is to be minimized.

No simple equations for the transition paths are found, but some information about them is discovered by the trial and error method.

The airplane selected for the calculations is a swept wing, single seat, present day interceptor type equipped with a single jet engine and afterburner. Calculations are made with the afterburner both operative and inoperative, since with the afterburner operating the climb performance is seriously limited by compressibility, while with afterburner inoperative the Mach numbers encountered in climb are below the drag divergence Mach number.

2. EQUATIONS OF MOTION



The symbols used in this report are as follows:

D = drag

L = lift

T = thrust

W = weight

F = centrifugal force

θ = angle of climb

R = radius of curvature

V = flight path velocity

S = distance along flight path

C = rate of climb

h = altitude

x = horizontal distance covered

$R_o = \left(\frac{T-D}{W}\right) V$ = energy available for climb and/or acceleration per unit weight

$n = \frac{L}{W}$ = load factor

f = fuel used

W_f = rate of fuel consumption

t = time

A prime denotes total differentiation with respect to h. Thus $V' = \frac{dV}{dh}$ etc.

The equations of motion for two dimensional flight in a vertical plane are:

$$T - D - W \sin \theta = \frac{W}{g} \frac{dV}{dt} \qquad 2.1$$

$$L - W \cos \theta = F \qquad 2.2$$

Note that

$$F = \frac{W}{g} \frac{V^2}{R} = \frac{W}{g} \frac{V^2}{\frac{dS}{d\theta}} = \frac{W}{g} \frac{V}{\frac{dt}{d\theta}} = \frac{W}{g} V \frac{d\theta}{dt}$$

Making use of this expression for F, the definitions of R_o and n, and the fact that $V \sin \theta = C$, 2.1 and 2.2 may be written as

$$C = \frac{R_o}{1 + \frac{V}{g} V'} \qquad 2.3$$

$$n = \cos \theta + \frac{V^2}{g} \theta' \sin \theta \qquad 2.4$$

3. APPROACH THROUGH THE CALCULUS OF VARIATIONS

In this section the equations defining the path of least time between specified initial and final speeds, altitudes, and angles of climb will be presented. The method used for finding the minimizing path is a modification of the method of Bolza and has been very completely described by M.R. Hestenes in Reference 1. Consequently no description of the method will be given here.

The problem may be stated as follows: It is desired to find n and V as functions of h such that the time

$$t = \int_{h_1}^{h_2} \frac{dh}{V \sin \theta}$$

will be a minimum when the following relations between θ, V, n, x, h hold at every point on the path (see equations 2.3 and 2.4).

$$\left.\begin{aligned} \frac{d\theta}{dh} &= \frac{g}{V^2} \frac{(n - \cos \theta)}{\sin \theta} \\ \frac{dV}{dh} &= \frac{g}{V} \left[\frac{R_o}{V \sin \theta} - 1 \right] \end{aligned}\right\} \qquad 3.1$$

$$\left.\begin{aligned}\frac{dx}{dh} &= \cot\theta \quad \text{(inclusion of this equation allows the distance covered to be specified).}\\ R_o &= R_o(V,h,n)\end{aligned}\right\} \quad 3.1$$

The procedure, as given in Reference 1 is to construct a function H such that

$$H = p_\theta \frac{g}{V^2}\frac{(n-\cos\theta)}{\sin\theta} + p_V \frac{g}{V}\left[\frac{R_o}{V\sin\theta} - 1\right] + p_x \cot\theta - \frac{1}{V\sin\theta}$$

where the p's are unknown functions of h. It is shown in Reference 1 that, on the minimizing path, the following relations must hold:

$$\frac{dp_x}{dh} = -\frac{\partial H}{\partial x}, \quad \frac{dp_\theta}{dh} = -\frac{\partial H}{\partial\theta}, \quad \frac{dp_V}{dh} = -\frac{\partial H}{\partial V}, \quad \frac{\partial H}{\partial n} = 0$$

These relations lead to

$$\left.\begin{aligned}\frac{dp_\theta}{dh} &= -\frac{1}{\sin^2\theta}\left[p_\theta \frac{g}{V^2}(1 - n\cos\theta) - p_V \frac{g}{V^2}R_o\cos\theta - p_x + \frac{\cos\theta}{V}\right]\\ \frac{dp_V}{dh} &= \frac{1}{V^2}\left[2p_\theta \frac{g}{V}\frac{(n-\cos\theta)}{\sin\theta} + p_V\, g\left(\frac{2R_o}{V\sin\theta} - 1 - \frac{1}{\sin\theta}\frac{\partial R_o}{\partial V}\right) - \frac{1}{\sin\theta}\right]\\ p_x &= \text{constant}\\ p_\theta &+ p_V \frac{\partial R_o}{\partial n} = 0\end{aligned}\right\} \quad 3.2$$

The equations 3.1 and 3.2, together with the specified initial and final values of V, x, θ, and h, will determine the minimizing path. Unfortunately, these equations are rather complicated and they do not provide much physical insight into the problem.

4. TRIAL AND ERROR APPROACH

In order to obtain some feeling for the problem, a number of arbitrary paths were assumed and the time and fuel used along them calculated. The time required in the climb portion of the flight was determined from equation 2.3 as

$$t = \int_{h_1}^{h_2} \frac{\left(1 + \frac{VV'}{g}\right)dh}{R_o}.$$

If the flight involved acceleration at constant altitude, the time to accelerate in level flight was calculated from equation 2.1 as

$$t = \int_{V_1}^{V_2} \frac{V}{gR_o} dV \ .$$

The fuel was calculated as

$$f = \int_{o}^{t_2} W_f \, dt$$

and the horizontal distance covered was calculated from

$$x = \int_{o}^{t_2} V \, dt$$

The drag, thrust, weights, and fuel flows were taken from the airplane manufacturers' performance report. The weight is varied along the flight path to allow approximately for the fuel used.

In making these calculations, R_o was evaluated by assuming $n = 1$ throughout the flight. This approximation was checked by calculating the time along path 4, Fig. 1, using the correct values of n as obtained from equation 2.4. It was found that the exact and approximate times differed by less than one percent. Since path 4 involves a more abrupt pull-up, and consequently a greater variation of n, than any of the other paths shown, it is felt that this check justified the use of the approximation along the other paths. The exact values of n along path 4 are shown in Fig. 3.

The results of the calculations to find the flight technique for minimum time with distance unrestricted are summarized in Figs. 1 and 2. Paths to reach three different speeds at 35,000 feet are shown with most desirable paths distinguished by heavy lines. Two conclusions can be drawn from Figs. 1 and 2. The first is that a large portion of the optimum path remains unchanged as the final velocity is varied. This immediately suggests the existence of a characteristic path of the type defined in section 1. The second conclusion is that reasonably large variations can be made from the optimum path without severe penalties in time.

The paths investigated to determine the techniques for minimum fuel with distance unrestricted are summarized in Fig. 4 where the most desirable paths are distinguished by heavy lines. Again it is seen that a characteristic path exists and that reasonably large variations from the optimum can be made without severe penalties in fuel.

The lower part of Fig. 4 shows the effect of engine operation on the fuel required to reach 500 mph at 35,000 feet. The path for each throttle setting is approximately that requiring the least fuel to reach the specified speed and altitude with the given throttle setting. It can be seen that operation at military power without afterburning requires the least fuel.

5. EQUATIONS OF THE CHARACTERISTIC PATHS FOR MINIMUM TIME

To find the equations of the characteristic paths, we note that the definition of these paths implies that the characteristic path has the property of minimizing the time between any two points which it connects. It follows that minute deviations from this path must not cause any change in the time required. Thus, it is seen that, along the characteristic path the value of integral representing the time used is independent of small variations in the path shape. The integral representing the time may be written as (see equation 2.3)

$$t = \int_{h_1,V_1}^{h_2,V_2} \frac{dh}{R_o} + \frac{V}{gR_o}\, dV$$

We shall make use of the approximation that R_o may be evaluated with $n = 1$, so that R_o may be considered as a function of V and h only. The condition that the time integral be independent of the path is given by

$$\frac{\partial}{\partial V}\left(\frac{1}{R_o}\right) = \frac{\partial}{\partial h}\left(\frac{V}{gR_o}\right)$$

or

$$\frac{V}{g}\,\frac{\partial R_o}{\partial h} - \frac{\partial R_o}{\partial V} = 0 \qquad 5.1$$

Since for a particular airplane, R_o is a given function of V and h, equation 5.1 describes a path of V vs. h which is the characteristic path for minimum time with distance unspecified.

The reasoning in finding the characteristic path for minimum time with distance specified is very similar. We wish to find the path along which the integral

$$t = \int_{h_1,\, V_1}^{h_2,V_2} \frac{dh}{R_o} + \frac{V}{gR_o}\, dV$$

is independent of small variations in the path under the condition that the distance,

$$x = \int_{h_1,V_1}^{h_2,V_2} \frac{V}{R_o}\, dh + \frac{V^2}{gR_o}\, dV$$

is constant. Since the only admissable variations are those for which the distance integral is independent of changes in the path, this problem is equivalent to finding the path along which the integral

$$\int_{h_1, V_1}^{h_2, V_2} \frac{(1 + \lambda V)}{R_o} dh + \frac{(1 + \lambda V)V}{gR_o} dV$$

is independent of small variations in path. The λ is a constant which determines the distance covered along the characteristic path for a specified change in altitude. The condition that this integral be independent of the path is

$$\frac{\partial}{\partial V}\left[\frac{(1 + \lambda V)}{R_o}\right] = \frac{\partial}{\partial h}\left[\frac{V}{g}\,\frac{(1 + \lambda V)}{R_o}\right]$$

or

$$\frac{\lambda R_o}{1 + \lambda V} + \frac{V}{g}\,\frac{\partial R_o}{\partial h} - \frac{\partial R_o}{\partial V} = 0 \qquad 5.2$$

This path has the property that the time between any two points on it is less along the characteristic path than it is along any other path connecting the given points and having the same value of the distance integral. The optimum flight path for minimum time with a total distance specified will consist of a transition to one of the characteristic paths defined by 5.2, a climb up the characteristic path, and a transition from the characteristic path to the final speed and altitude. The value of λ used to select the exact characteristic path must be determined so that the distance covered along the characteristic path will be equal to the specified distance minus the distance covered on the transition paths.

The characteristic path involving the highest speeds is obtained by letting $\lambda \rightarrow \infty$, in which case equation 5.2 becomes

$$\frac{R_o}{V} + \frac{V}{g}\,\frac{\partial R_o}{\partial h} - \frac{\partial R_o}{\partial V} = 0. \qquad 5.3$$

This path has the property that it minimizes the distance covered in moving between any two speed altitude points which it connects. Fig. 5 indicates that when the specified distance is long enough to allow the airplane's maximum speed in level flight to be attained, the characteristic path closely approaches equation 5.3. It will be noted, however, that the characteristic path defined by equation 5.1 may be used with only a very slight loss in time. Consequently, it appears that it is not necessary to use great care in the selection of the proper characteristic path from those defined by equation 5.2

Equation 5.2 will determine characteristic paths for use when the specified distance is less than that covered along the path for minimum time with distance unrestricted. These paths, however, are not of much practical value, since the technique on such a flight would be to follow the path for minimum time with distance unrestricted and turn slowly during the flight to adjust the distance to specified value.

Equations 5.1 and 5.3 define a band of speeds in which the majority of climbing should be done. This band is shown in Fig. 6 for a typical modern interceptor operating at military thrust and at military thrust plus afterburning. The effect of compressibility is clearly evident in the narrowness of the band with afterburner operating compared to the band with afterburner inoperative.

6. EQUATIONS OF CHARACTERISTIC PATHS FOR MINIMUM FUEL

It was found in Section 4 that the best engine operation to obtain minimum fuel with distance unrestricted is to use military thrust without afterburning throughout the flight. The fact that a fixed throttle setting is used allows R_o (for n = 1) and W_f to be represented as functions of speed and altitude only. Consequently, the characteristic path for minimum fuel with distance unrestricted can be found in the same manner as that for minimum time, distance unrestricted. The equation is found to be

$$\frac{W_f}{R_o}\left(\frac{\partial R_o}{\partial V} - \frac{V}{g}\frac{\partial R_o}{\partial h}\right) - \left(\frac{\partial W_f}{\partial V} - \frac{V}{g}\frac{\partial W_f}{\partial h}\right) = 0 \qquad 6.1$$

A similar equation can be obtained for the characteristic path for minimum fuel, distance specified, provided that the entire flight is made at a fixed throttle setting. The equation is found to be

$$\frac{(W_f + \lambda V)}{R_o}\left(\frac{\partial R_o}{\partial V} - \frac{V}{g}\frac{\partial R_o}{\partial h}\right) - \left(\frac{\partial W_f}{\partial V} - \frac{V}{g}\frac{\partial W_f}{\partial h}\right) - \lambda = 0 \qquad 6.2$$

This equation is not of great practical use, however, since the best technique for minimum fuel with a specified distance greater than the distance covered in a military power climb involves operation at part throttle.

7. TRANSITION PATHS

No formulas have been developed to describe the transition paths. A series of transition paths from take-off speed to the characteristic path for minimum time with distance unrestricted is shown in Fig. 7. It can be seen that although the difference in time between the better paths is small, some time can be saved by diving during the transition. Ordinarily, it will not be possible to dive immediately after take off, so that acceleration at constant altitude will be the best transition path.

The best transition technique from the characteristic path to the final speed and altitude depends on whether the final speed is greater or less than the speed on the characteristic path at the final altitude. If it is less, the best procedure is to leave the characteristic path at an altitude below the final altitude and zoom to the final speed. This technique is illustrated by paths 4 and 6 in Fig. 1 and by path 11 in Fig. 2. If the final speed is greater than that on the characteristic path at the final altitude, the curves of Fig. 7 indicate that the best technique is to climb above the final altitude and dive to the final speed and altitude. Although the time saved by diving instead of accelerating in level flight is small, there may be a considerable tactical advantage in diving.

8. CONCLUSIONS

The conclusions may be summarized as follows:

1. If the final altitude is sufficiently far above the initial, the greater portion of the flight paths which minimize time or fuel will lie along a speed-altitude path which is independent of the specified intial and final speeds and altitudes. This path is called the "characteristic path". The optimum flight path consists of a transition from the initial speed and altitude to the characteristic path, a climb along this path, and a transition from the characteristic path to the final speed and altitude.

2. The characteristic path depends on the specified distance to be covered in the flight, the throttle setting used during the flight, and the design of the airplane.

3. The characteristic paths for distance unrestricted and for large specified distances form a band of speeds in which the majority of climbing should be done.

4. Simple formulas for calculating the characteristic paths which determine the climbing bands are given in Sections 5 and 6.

5. The best transition procedure involves diving if an increase in speed is necessary and zooming if a decrease in speed is necessary. If diving is impractical, very little time is lost by accelerating at constant altitude.

6. Reasonably large departures from the true minimizing paths may be made with only small losses in time or fuel.

7. The best engine operation for minimum fuel if distance is unrestricted is to use military thrust without afterburning.

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FIG. 1

COMPARISON OF FLIGHT TIMES ALONG VARIOUS PATHS
FOR A TYPICAL PRESENT DAY INTERCEPTOR.
DISTANCE UNRESTRICTED
MILITARY THRUST + AFTERBURNER.

PATH	TIME (MIN)
1	3.94
2	3.13
3	2.46
4	2.31

PATH	TIME (MIN)
5	2.47
6	2.41

PATH	TIME (MIN)
7	2.61
8	2.63

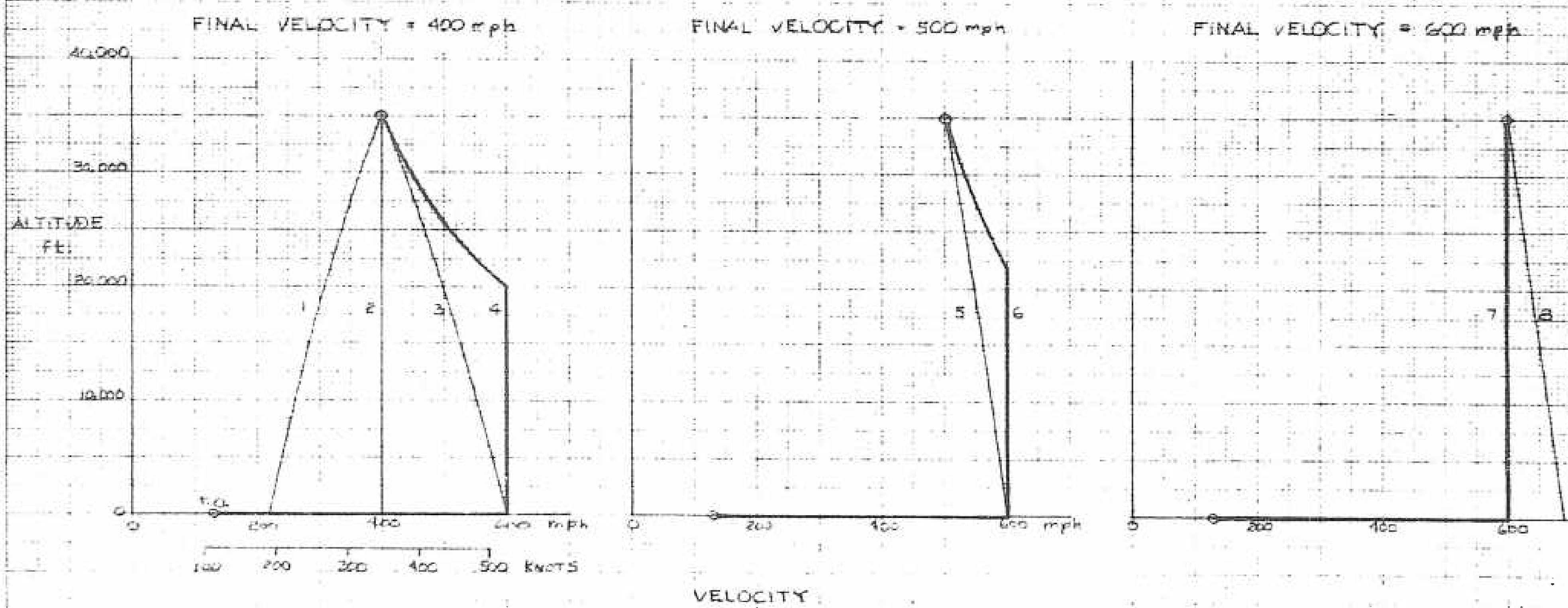

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FIG. 2

COMPARISON OF FLIGHT TIMES ALONG VARIOUS PATHS
FOR A TYPICAL PRESENT DAY INTERCEPTOR.
DISTANCE UNRESTRICTED
MILITARY THRUST - AFTERBURNER INOPERATIVE.

PATH	TIME (MIN)
9	6.60
10	6.38
11	6.15

PATH	TIME (MIN)
12	6.81
13	6.87
14	7.10

PATH	TIME (MIN)
15	8.03
16	8.25
17	8.44

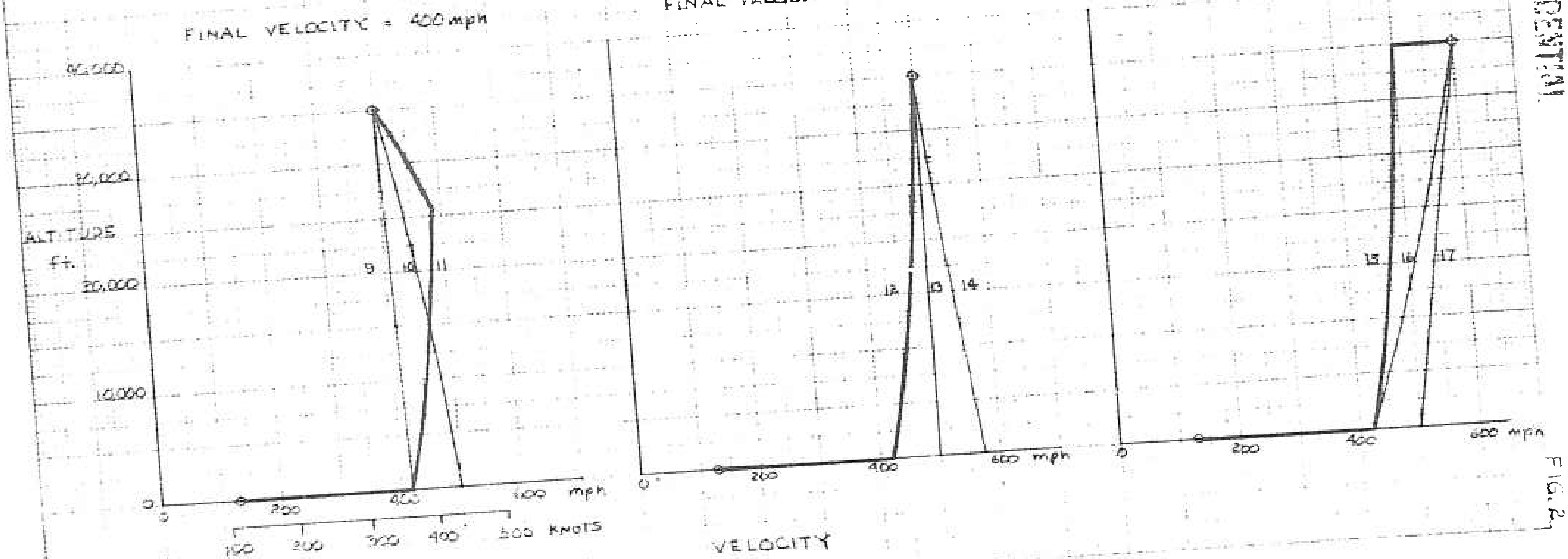

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FIG. 3

VARIATION OF LOAD FACTOR WITH ALTITUDE ALONG PATH 4.

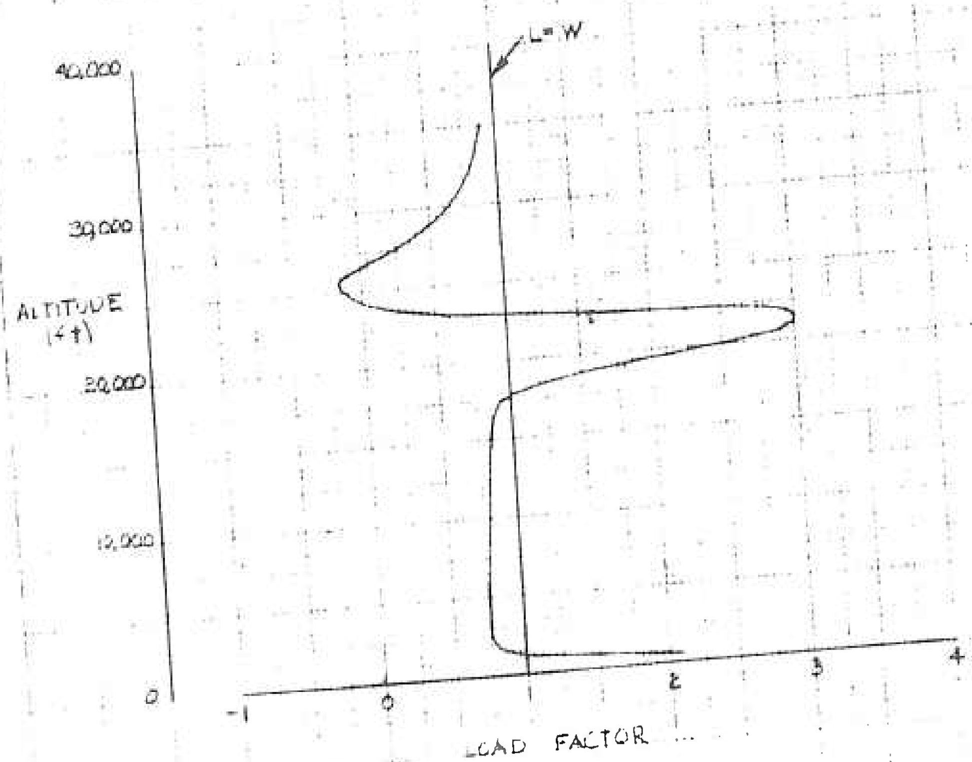

FIG. 4

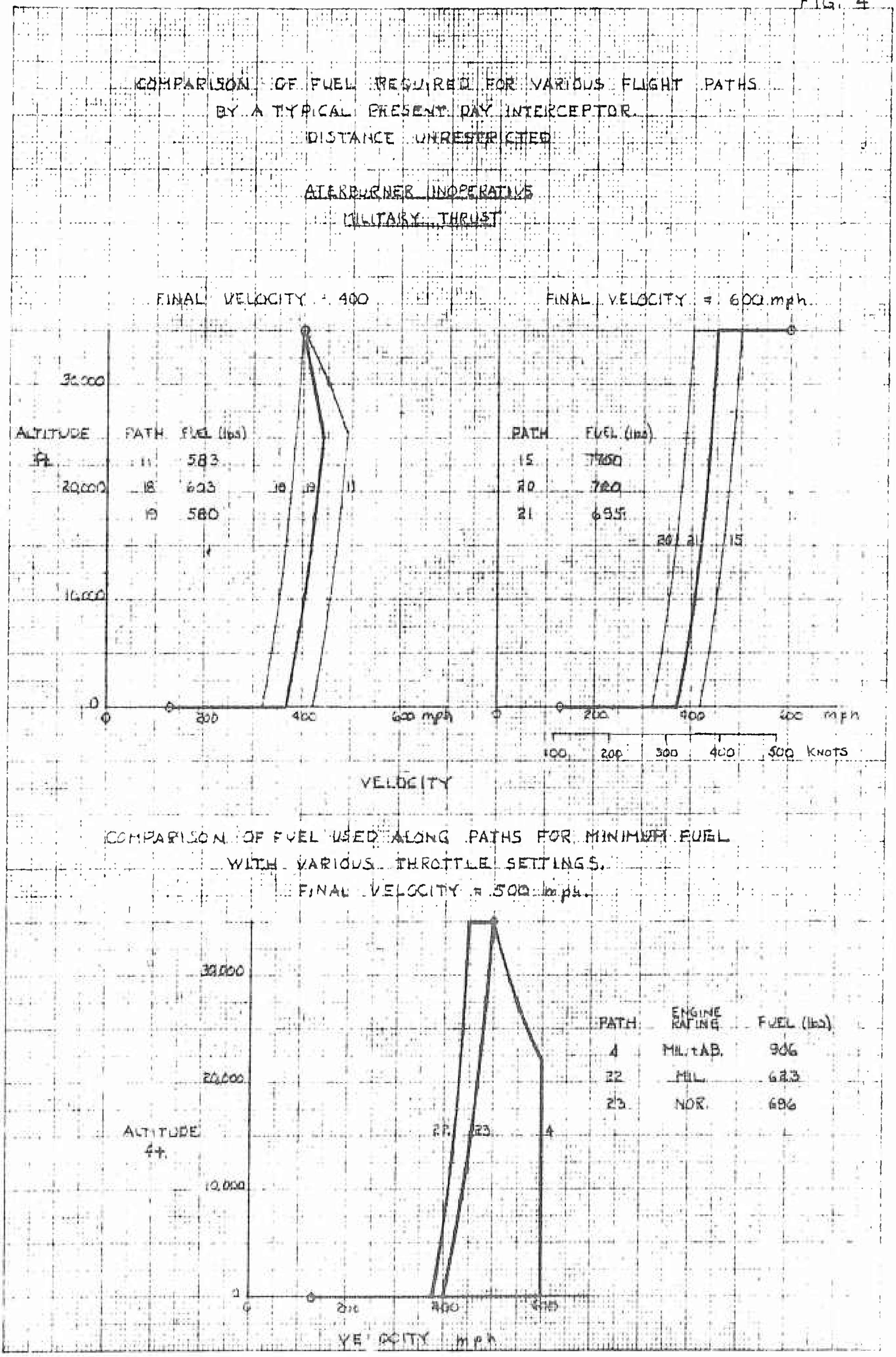
COMPARISON OF FUEL REQUIRED FOR VARIOUS FLIGHT PATHS
BY A TYPICAL PRESENT DAY INTERCEPTOR.
DISTANCE UNRESTRICTED
AFTERBURNER INOPERATIVE
MILITARY THRUST
FINAL VELOCITY = 400
FINAL VELOCITY = 600 mph.
30,000
20,000
10,000
0
ALTITUDE
Ft.
PATH FUEL (lbs)
11 583
18 603
19 580
18 19 11
PATH FUEL (lbs)
15
20 720
21 695
20 21 15
0 200 400 600 mph
0 200 400 600 mph
100 200 300 400 500 KNOTS
VELOCITY
COMPARISON OF FUEL USED ALONG PATHS FOR MINIMUM FUEL
WITH VARIOUS THROTTLE SETTINGS.
FINAL VELOCITY = 500 mph.
30,000
20,000
10,000
0
ALTITUDE
Ft.
PATH ENGINE RATING FUEL (lbs)
4 MIL+AB. 906
22 MIL. 683
23 NOR. 696
22 23 4
0 200 400
VELOCITY mph

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FIG. 5

COMPARISON OF FLIGHT TIMES ALONG VARIOUS PATHS
FOR A TYPICAL PRESENT DAY INTERCEPTOR
MILITARY THRUST, AFTERBURNER
INOPERATIVE.
DISTANCE SPECIFIED.

ALL PATHS COVER A DISTANCE OF 150 mi.

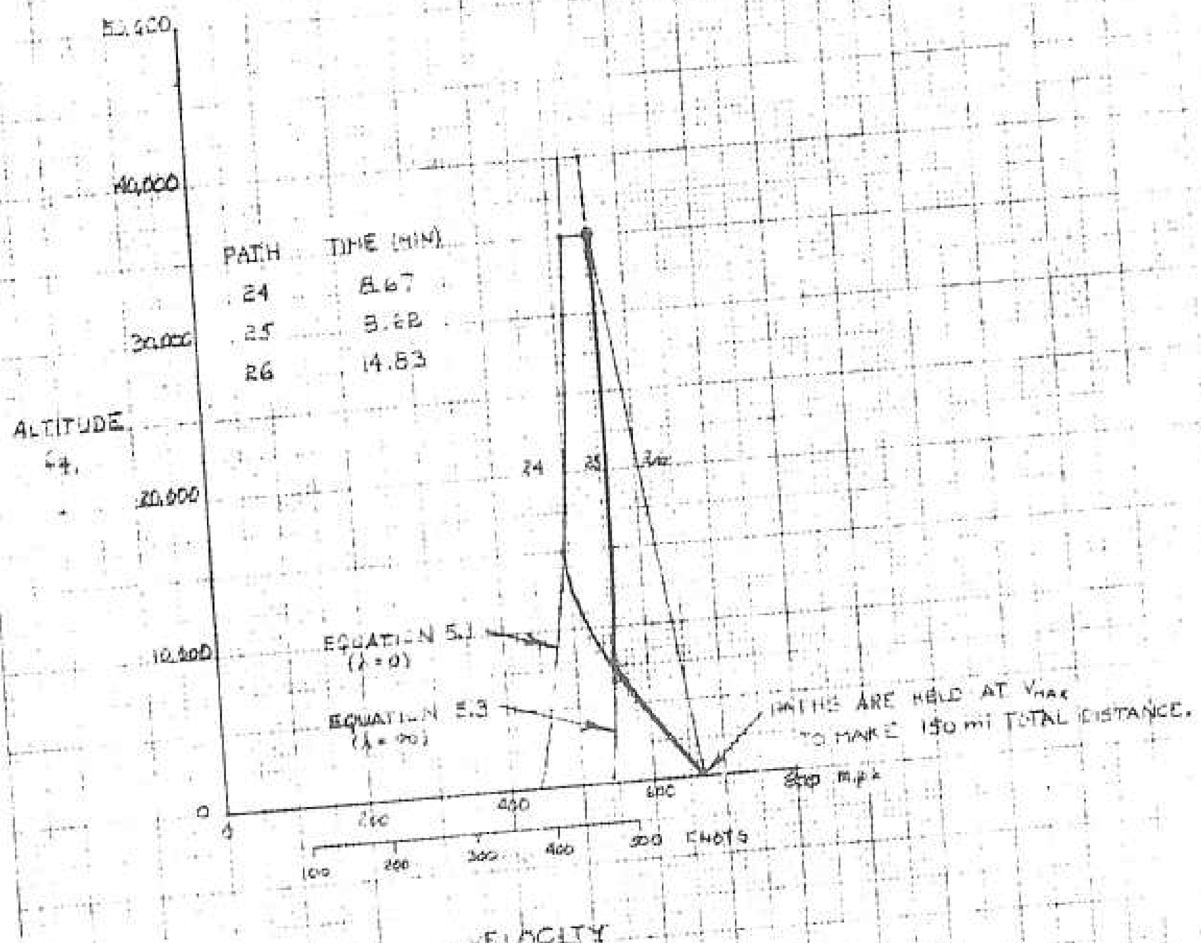

FIG. 8

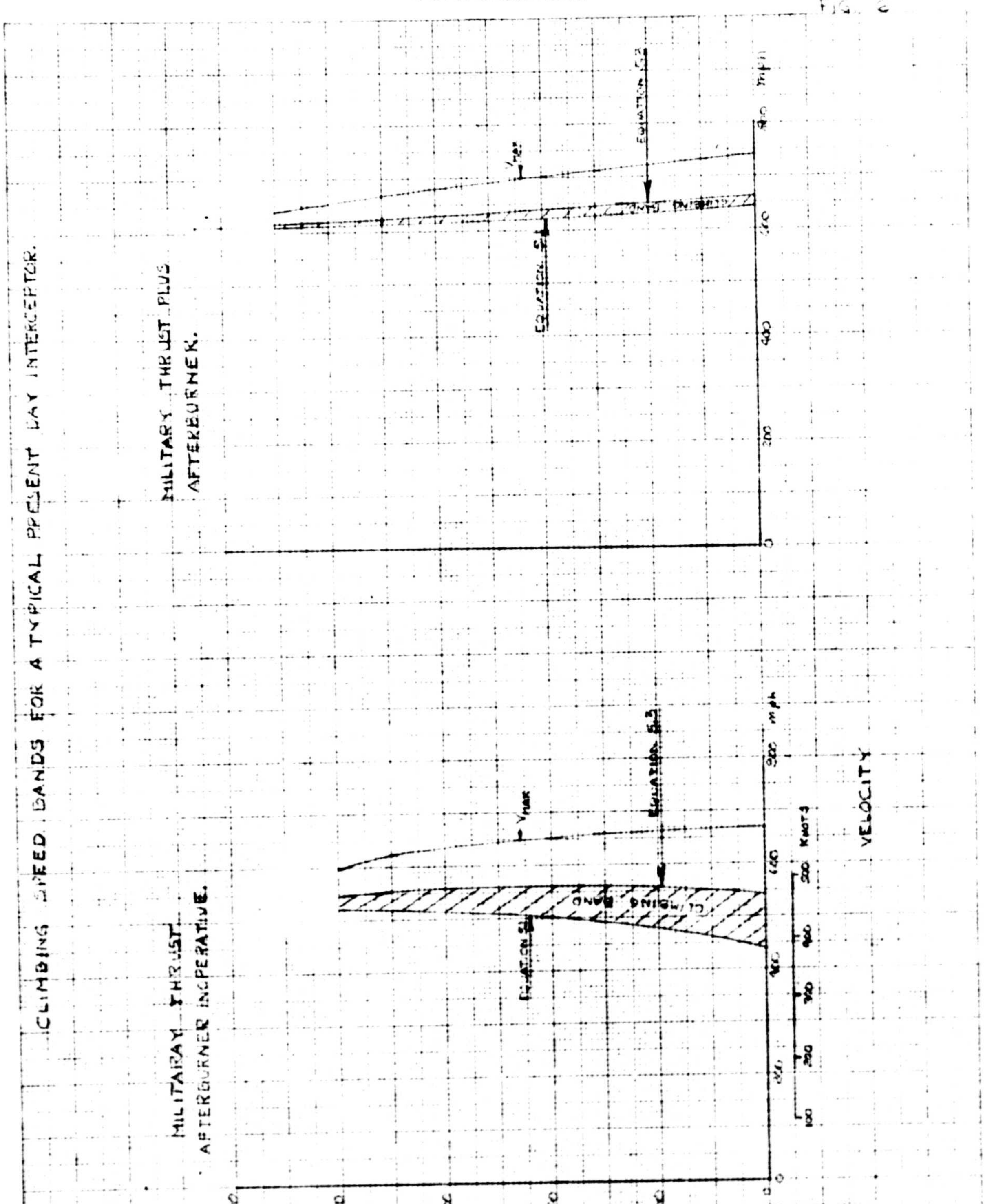
CLIMBING SPEED BANDS FOR A TYPICAL PRESENT DAY INTERCEPTOR.
MILITARY THRUST PLUS AFTERBURNER.
MILITARY THRUST AFTERBURNER INOPERATIVE.
V_{max}
EQUATION 5.3
EQUATION 5.1
CLIMBING BAND
mph
KNOTS
VELOCITY
ALTITUDE ft

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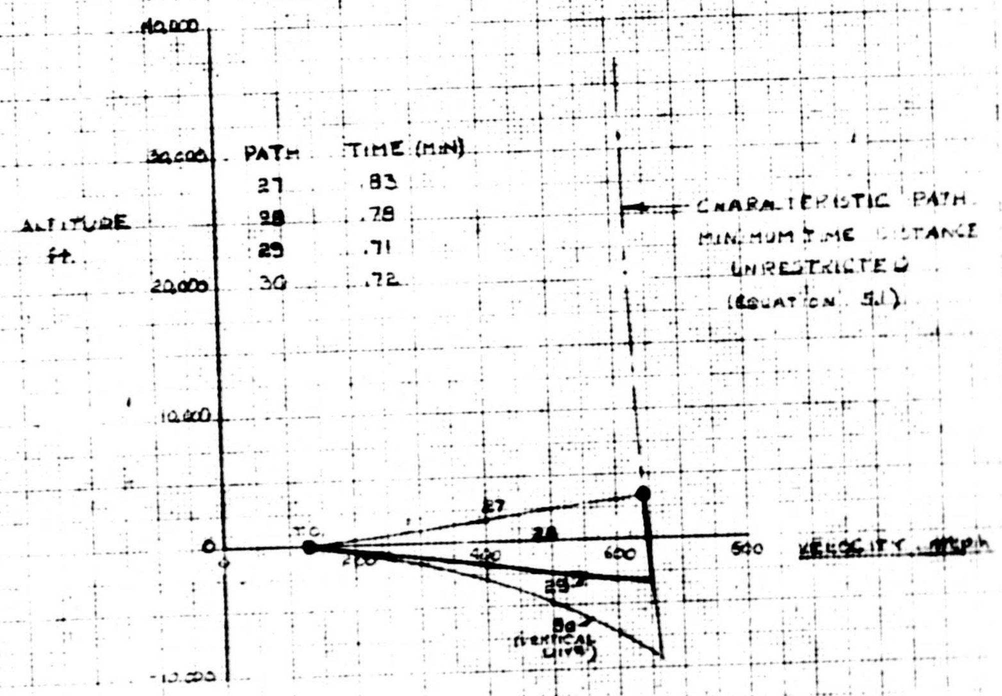
FIG. 7

COMPARISON OF TIMES ALONG VARIOUS TRANSITION PATHS FOR A TYPICAL PRESENT DAY INTERCEPTOR.

MILITARY THRUST PLUS AFTERBURNER



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